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2002 J. Phys. A: Math. Gen. 35 L283

(http://iopscience.iop.org/0305-4470/35/21/102)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) L283–L291 PII: S0305-4470(02)34769-3

LETTER TO THE EDITOR

Integration of the modified Korteweg–de Vries hierarchy with an integral type of source

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Received 8 March 2002
Published 17 May 2002
Online at stacks.iop.org/JPhysA/35/L283

Abstract

The modified Korteweg–de Vries hierarchy with an integral type of source (mKdVHWS), which consists of the reduced AKNS eigenvalue problem with $r = q$ and the mKdV hierarchy with an extra term of the integration of a square eigenfunction, is investigated. We propose a method to find the explicit evolution equation for the eigenfunction of the auxiliary linear problems of the mKdVHWS. Then we determine the evolution equations of scattering data corresponding to the mKdVHWS, and solve the equation in the mKdVHWS by inverse scattering transformation.

PACS numbers: 02.30.Ik, 05.45.Yv

1. Introduction

The nonlinear Schrödinger equation with an integral type of source (NLSEWS) is relevant to some problems of plasma physics and solid-state physics [1]. The NLSEWS in some case was studied by so called $\bar{\partial}$-method in [1], and it was stated that this NLSEWS could not be integrated by the classical inverse scattering method. Later it was shown in [2] that the NLSEWS can be integrated by the inverse scattering method for the Dirac operator. The key point of the application of the inverse scattering method to integration of the NLSEWS in [2] is the use of the determining relations playing the same role as different operator representations of the Lax type of nonlinear evolution equation integrable by various modifications of this method. Just using the determining relations Mel'nikov obtained the evolution equations for all the scattering data of the Dirac operator corresponding to NLSEWS. A similar method was used to investigate the Korteweg–de Vries equation with an integral type of source (KdVWS) in [3]. The reason for the use of the determining relations in [2, 3] is that the evolution equation of the eigenfunction for the eigenvalue problem corresponding to the NLSEWS and KdVWS was not found. In fact, the establishment of these determining relations and the derivation of the evolution equations for all scattering data in [2, 3] are quite complicated and require some skill.

In this letter we investigate the new modified Korteweg–de Vries hierarchy with an integral type of source (mKdVHWS), which consists of the reduced AKNS eigenvalue problem with $r = q$ and the mKdV hierarchy with an extra term of the integration of a square eigenfunction. We first present a method to construct the zero-curvature representation for the mKdVHWS by finding the explicit evolution equation for the eigenfunction of the auxiliary linear problem for the mKdVHWS. Then we present a way to determine the evolution equation for the scattering data corresponding to the mKdVHWS, which implies that the mKdVHWS can be integrated by the inverse scattering method. Compared with the method using the determining relation in [2, 3], the method proposed in this letter for determining the evolution equation of the scattering data is quite natural and simple. This general method can be applied to other $(1+1)$-dimensional soliton equations with an integral type of source.

2. The mKdV hierarchy with an integral type of source

Consider the reduced AKNS eigenvalue problem for $r = q$ [4]

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x = U\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \qquad U = \begin{pmatrix}-\lambda & q\\ q & \lambda\end{pmatrix}. \tag{2.1}$$

The adjoint representation of (2.1) reads [5]

$$V_x = [U, V] = UV - VU. \tag{2.2}$$

Set

$$V = \sum_{i=0}^{\infty}\begin{pmatrix}a_i & b_i\\ c_i & -a_i\end{pmatrix}\lambda^{-i}. \tag{2.3}$$

Equation (2.2) yields

$$\begin{array}{lll} a_0 = -1, & b_0 = c_0 = a_1 = 0, & b_1 = c_1 = q\\ a_2 = \frac{1}{2}q^2, & b_2 = -c_2 = -\frac{1}{2}q_x, & \ldots, \end{array}$$

and in general

$$\begin{array}{ll} b_{2m+1} = c_{2m+1} = Lb_{2m-1}, & b_{2m} = -c_{2m} = -\frac{1}{2}Db_{2m-1},\\ a_{2m+1} = 0, & a_{2m} = 2D^{-1}qb_{2m} \end{array} \tag{2.4}$$

where

$$L = \frac{1}{4}D^2 - qD^{-1}qD, \qquad D = \frac{\partial}{\partial x}, \qquad DD^{-1} = D^{-1}D = 1.$$

Set

$$V^{(2n+1)} = \sum_{i=0}^{2n+1}\begin{pmatrix}a_i & b_i\\ c_i & -a_i\end{pmatrix}\lambda^{2n+1-i} \tag{2.5}$$

and take

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_{t_{2n+1}} = V^{(2n+1)}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}. \tag{2.6}$$

Then the compatibility conditions of equations (2.1) and (2.6) give rise to the mKdV hierarchy [4]

$$q_{t_{2n+1}} = -2b_{2n+2} = Db_{2n+1} = D\frac{\delta H_{2n+1}}{\delta q}, \qquad n = 0, 1, \ldots, \tag{2.7}$$

where

$$H_{2n+1} = \frac{2a_{2n+2}}{2n+1}.$$

Using (2.1), we have

$$\frac{\delta\lambda}{\delta q} = \phi_1^2 - \phi_2^2, \qquad L(\phi_1^2 - \phi_2^2) = \lambda^2(\phi_1^2 - \phi_2^2). \tag{2.8}$$

As proposed in [2, 3, 7, 8], the mKdV hierarchy with integral type of source is defined by

$$q_{t_{2n+1}} = D\left[b_{2n+1} + \int_{-\infty}^{\infty} C(t,\zeta)(\phi_1^2(x,t,\zeta) - \phi_2^2(x,t,\zeta))\,\mathrm{d}\zeta\right] \tag{2.9a}$$

$$\phi_{1,x} = -\mathrm{i}\zeta\phi_1 + q\phi_2, \qquad \phi_{2,x} = q\phi_1 + \mathrm{i}\zeta\phi_2 \qquad \zeta \in (-\infty, \infty); \tag{2.9b}$$

we assume $q(x, t_{2n+1})$ tends rather quickly to zero as $x \to \pm\infty$. According to this condition we assume that

$$\phi_1(x,t,\zeta) \sim a\exp(-\mathrm{i}\zeta x), \qquad \phi_2(x,t,\zeta) \sim b\exp(\mathrm{i}\zeta x), \qquad x \to -\infty \tag{2.10}$$

where $C = C(t,\zeta)$, $a = a(t,\zeta)$ and $b = b(t,\zeta)$ are complex functions of $t \geqslant 0$ and $\zeta \in (-\infty, \infty)$. Moreover we assume that the functions C, a and b are chosen so that the right-hand side of equation (2.9) determines the function absolutely integrable over x along the whole real axis. One can easily verify that the requirement will certainly be satisfied if the functions E and Γ of the form argued in [2]

$$E = |C(t,\zeta)|[|a(t,\zeta)| + |b(t,\zeta)|]^2 \qquad \Gamma = \left|\frac{\partial}{\partial\zeta}[C(t,\zeta)a^2(t,\zeta)]\right| + \left|\frac{\partial}{\partial\zeta}[C(t,\zeta)b^2(t,\zeta)]\right|$$

at any $t \geqslant 0$ satisfy the condition

$$\int_{-\infty}^{\infty}[E(t,\zeta) + \Gamma(t,\zeta) + \Gamma^2(t,\zeta)]\,\mathrm{d}\zeta < \infty.$$

3. The Lax representation

Following the method proposed in [6–8], in order to find the zero-curvature representation for (2.9), we first consider

$$D\left[b_{2n+1} + \int_{-\infty}^{\infty} C(t,\zeta)(\phi_1^2(x,t,\zeta) - \phi_2^2(x,t,\zeta))\,\mathrm{d}\zeta\right] = 0 \tag{3.1a}$$

$$\phi_{1,x} = -\mathrm{i}\zeta\phi_1 + q\phi_2, \qquad \phi_{2,x} = q\phi_1 + \mathrm{i}\zeta\phi_2 \qquad \zeta \in (-\infty, \infty). \tag{3.1b}$$

We can obtain the Lax representation for (3.1) by using the adjoint representation (2.2). According to (2.4), (2.8) and (3.1), we may define

$$\tilde{a}_i = a_i, \qquad \tilde{b}_i = b_i, \qquad \tilde{c}_i = c_i, \qquad i = 0, 1, \ldots, 2n,$$

$$\tilde{b}_{2n+2m+1} = \tilde{c}_{2n+2m+1} = L\tilde{b}_{2n+2m-1} = -\int_{-\infty}^{\infty}(\mathrm{i}\zeta)^{2m}C(t,\zeta)[\phi_1^2(x,t,\zeta) - \phi_2^2(x,t,\zeta)]\,\mathrm{d}\zeta$$

$$\tilde{b}_{2n+2m+2} = -\tilde{c}_{2n+2m+2} = -\tfrac{1}{2}D\tilde{b}_{2n+2m+1} = -\int_{-\infty}^{\infty}(\mathrm{i}\zeta)^{2m+1}C(t,\zeta)[\phi_1^2(x,t,\zeta) + \phi_2^2(x,t,\zeta)]\,\mathrm{d}\zeta$$

$$\tilde{a}_{2n+2m+2} = 2D^{-1}q\tilde{b}_{2n+2m+2} = -2\int_{-\infty}^{\infty}(\mathrm{i}\zeta)^{2m+1}C(t,\zeta)\phi_1(x,t,\zeta)\phi_2(x,t,\zeta)\,\mathrm{d}\zeta$$

$$\tilde{a}_{2n+2m+1} = 0, \qquad m = 0, 1, \ldots.$$

Then

$$N^{(2n+1)} = \begin{pmatrix} A^{(2n+1)} & B^{(2n+1)} \\ C^{(2n+1)} & D^{(2n+1)} \end{pmatrix} \equiv \lambda^{2n+1} \sum_{k=0}^{\infty} \begin{pmatrix} \tilde{a}_k & \tilde{b}_k \\ \tilde{c}_k & -\tilde{a}_k \end{pmatrix} \lambda^{-k} + \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}$$

where θ is some constant, and

$$A^{(2n+1)} = \sum_{k=0}^{2n} a_k \lambda^{2n+1-k} + \theta + \int_{-\infty}^{\infty} \frac{2(\mathrm{i}\zeta)(\mathrm{i}\eta)C(t,\eta)\phi_1(x,t,\eta)\phi_2(x,t,\eta)}{(\mathrm{i}\zeta)^2 - (\mathrm{i}\eta)^2}\,\mathrm{d}\eta$$

$$B^{(2n+1)} = \sum_{k=0}^{2n} b_k \lambda^{2n+1-k} + \int_{-\infty}^{\infty} \frac{\mathrm{i}\zeta(\mathrm{i}\zeta - \mathrm{i}\eta)C(t,\eta)\phi_2^2(x,t,\eta) - \mathrm{i}\zeta(\mathrm{i}\zeta + \mathrm{i}\eta)C(t,\eta)\phi_1^2(x,t,\eta)}{(\mathrm{i}\zeta)^2 - (\mathrm{i}\eta)^2}\,\mathrm{d}\eta$$

$$C^{(2n+1)} = \sum_{k=0}^{2n} c_k \lambda^{2n+1-k} + \int_{-\infty}^{\infty} \frac{\mathrm{i}\zeta(\mathrm{i}\zeta + \mathrm{i}\eta)C(t,\eta)\phi_2^2(x,t,\eta) - \mathrm{i}\zeta(\mathrm{i}\zeta - \mathrm{i}\eta)C(t,\eta)\phi_1^2(x,t,\eta)}{(\mathrm{i}\zeta)^2 - (\mathrm{i}\eta)^2}\,\mathrm{d}\eta$$

$$D^{(2n+1)} = -\sum_{k=0}^{2n} a_k \lambda^{2n+1-k} + \theta - \int_{-\infty}^{\infty} \frac{2(\mathrm{i}\zeta)(\mathrm{i}\eta)C(t,\eta)\phi_1(x,t,\eta)\phi_2(x,t,\eta)}{(\mathrm{i}\zeta)^2 - (\mathrm{i}\eta)^2}\,\mathrm{d}\eta$$

also satisfies the adjoint representation (2.2), i.e.

$$N_x^{(2n+1)} = [U, N^{(2n+1)}], \tag{3.2}$$

which, in fact, gives rise to the Lax representation of (3.1). Since (3.1) is the stationary equation of (2.9), it is easy to find that the zero-curvature representation for the mKdVHWS (2.9) is given by

$$U_{t_{2n+1}} - N_x^{(2n+1)} + [U, N^{(2n+1)}] = 0, \tag{3.3}$$

with the auxiliary linear problems

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} -\lambda & q \\ q & \lambda \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} -\mathrm{i}\zeta & q \\ q & \mathrm{i}\zeta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{3.4a}$$

where $\lambda = \mathrm{i}\zeta$ and

$$\begin{aligned} \psi_{1,t_{2n+1}} &= (A^{(2n+1)} + \theta)\psi_1 + B^{(2n+1)}\psi_2 \\ &\equiv \sum_{k=0}^{k=2n} (a_k\psi_1 + b_k\psi_2)\lambda^{2n+1-k} + \theta\psi_1 \\ &\quad + \int_{-\infty}^{\infty} \frac{\mathrm{i}\zeta C(t,\eta)}{(\mathrm{i}\zeta)^2 - (\mathrm{i}\eta)^2}[2(\mathrm{i}\eta)\phi_1(x,t,\eta)\phi_2(x,t,\eta)\psi_1 \\ &\quad + (\mathrm{i}\zeta - \mathrm{i}\eta)\phi_2^2(x,t,\eta)\psi_2 - (\mathrm{i}\zeta + \mathrm{i}\eta)\phi_1^2(x,t,\eta)\psi_2]\,\mathrm{d}\eta, \\ \psi_{2,t_{2n+1}} &= C^{(2n+1)}\psi_1 + (-A^{(2n+1)} + \theta)\psi_2 \\ &\equiv \sum_{k=0}^{k=2n} (c_k\psi_1 - a_k\psi_2)\lambda^{2n+1-k} + \theta\psi_2 \\ &\quad + \int_{-\infty}^{\infty} \frac{\mathrm{i}\zeta C(t,\eta)}{(\mathrm{i}\zeta)^2 - (\mathrm{i}\eta)^2}[(\mathrm{i}\zeta + \mathrm{i}\eta)\phi_2^2(x,t,\eta)\psi_1 - (\mathrm{i}\zeta - \mathrm{i}\eta)\phi_1^2(x,t,\eta)\psi_1 \\ &\quad - 2(\mathrm{i}\eta)\phi_1(x,t,\eta)\phi_2(x,t,\eta)\psi_2]\,\mathrm{d}\eta. \end{aligned} \tag{3.4b}$$

In this way we find the explicit evolution equations of eigenfunction ψ. Indeed, this kind of evolution equation of eigenfunction was not obtained in [2, 3].

4. Evolution equation for the reflection coefficients

Now we can derive equations describing the evolution in time t of the S-matrix elements. This can be performed as follows. We define the eigenfunctions $f^-(x,\zeta) = (f_1^-(x,\zeta), f_2^-(x,\zeta))^T$, $\bar{f}^-(x,\zeta) = (\bar{f}_1^-(x,\zeta), \bar{f}_2^-(x,\zeta))^T$, $f^+(x,\zeta) = (f_1^+(x,\zeta), f_2^+(x,\zeta))^T$ and $\bar{f}^+(x,\zeta) = (\bar{f}_1^+(x,\zeta), \bar{f}_2^+(x,\zeta))^T$ (here and hereafter 'T' means transposition) for equation (3.4a), and the following asymptotics are fulfilled at any $\zeta \in (-\infty,\infty)$:

$$f^-(x,\zeta) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{-\mathrm{i}\zeta x}, \qquad \bar{f}^-(x,\zeta) \sim \begin{pmatrix} 0 \\ -1 \end{pmatrix} \mathrm{e}^{\mathrm{i}\zeta x}, \qquad \text{as } x \to -\infty \tag{4.1a}$$

$$f^+(x,\zeta) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{e}^{\mathrm{i}\zeta x}, \qquad \bar{f}^+(x,\zeta) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{-\mathrm{i}\zeta x}, \qquad \text{as } x \to +\infty. \tag{4.1b}$$

As is known, the functions $f^-(x,\zeta)$ and $f^+(x,\zeta)$ admit an analytical continuation in the parameter ζ into the upper half-plane $\mathrm{Im}\,\zeta > 0$, and the functions $\bar{f}^-(x,\zeta)$ and $\bar{f}^+(x,\zeta)$ admit an analytical continuation in the parameter ζ into the lower half-plane $\mathrm{Im}\,\zeta < 0$. It is easily seen that at any real $\zeta \in (-\infty,\infty)$ the pair of functions $f^-(x,\zeta)$ and $\bar{f}^-(x,\zeta)$ forms a fundamental system of solutions to (3.4a). Hence, we may define

$$f^+(x,\zeta) = S_{12}(\zeta)\bar{f}^-(x,\zeta) + S_{22}(\zeta)f^-(x,\zeta) \tag{4.2a}$$

$$\bar{f}^+(x,\zeta) = S_{11}(\zeta)\bar{f}^-(x,\zeta) + S_{21}(\zeta)f^-(x,\zeta) \tag{4.2b}$$

where the quantities $S_{11} = S_{11}(\zeta)$, $S_{12} = S_{12}(\zeta)$, $S_{21} = S_{21}(\zeta)$ and $S_{22} = S_{22}(\zeta)$ are independent of x. Taking account of (4.1) and (4.2), we obtain at any $\zeta \in (-\infty,\infty)$ the equality

$$S_{11}(\zeta)S_{22}(\zeta) - S_{12}(\zeta)S_{21}(\zeta) = 1. \tag{4.3}$$

Under the assumption that $q(x,t)$ vanishes rapidly as $|x| \to \infty$, we have

$$a_0 = -1, \qquad b_0 = c_0 = 0, \qquad \lim_{|x|\to\infty} a_j = \lim_{|x|\to\infty} b_j = \lim_{|x|\to\infty} c_j = 0,$$
$$j = 1, 2, \ldots, 2n.$$

We denote the parameter θ in (3.4b) corresponding to $f^+(x,\zeta)$ by θ^+ and $\bar{f}^+(x,\zeta)$ by $\bar{\theta}^+$, respectively. Substituting $f^+(x,\zeta)$ and $\bar{f}^+(x,\zeta)$ into (3.4b), we have

$$\begin{aligned}
\frac{\partial f_1^+(x,\zeta)}{\partial t_{2n+1}} &= \Bigg\{\sum_{k=0}^{2n} a_k(\mathrm{i}\zeta)^{2n+1-k} + \theta^+ - \oint_{-\infty}^{\infty}\left(\frac{\zeta}{\eta-\zeta} + \frac{\zeta}{\eta+\zeta}\right)H(\eta)\,\mathrm{d}\eta \\
&\quad - \pi(\mathrm{i}\zeta)H(\zeta) + \pi(\mathrm{i}\zeta)H(-\zeta)\Bigg\} f_1^+(x,\zeta) \\
&\quad + \Bigg\{\sum_{k=0}^{2n} b_k(\mathrm{i}\zeta)^{2n+1-k} + \oint_{-\infty}^{\infty}\frac{\zeta}{\eta-\zeta}H_1(\eta)\,\mathrm{d}\eta + \oint_{-\infty}^{\infty}\frac{\zeta}{\eta+\zeta}H_2(\eta)\,\mathrm{d}\eta \\
&\quad + \pi(\mathrm{i}\zeta)H_1(\zeta) - \pi(\mathrm{i}\zeta)H_2(-\zeta)\Bigg\} f_2^+(x,t,\zeta),
\end{aligned} \tag{4.4a}$$

$$\begin{aligned}
\frac{\partial f_2^+(x,\zeta)}{\partial t_{2n+1}} &= \Bigg\{\sum_{k=0}^{2n} c_k(\mathrm{i}\zeta)^{2n+1-k} - \oint_{-\infty}^{\infty}\frac{\zeta}{\eta-\zeta}H_2(\eta)\,\mathrm{d}\eta - \oint_{-\infty}^{\infty}\frac{\zeta}{\eta+\zeta}H_1(\eta)\,\mathrm{d}\eta \\
&\quad - \pi(\mathrm{i}\zeta)H_2(\zeta) + \pi(\mathrm{i}\zeta)H_1(-\zeta)\Bigg\} f_1^+(x,\zeta) \\
&\quad + \Bigg\{\sum_{k=0}^{2n} -a_k(\mathrm{i}\zeta)^{2n+1-k} - \oint_{-\infty}^{\infty}\left(\frac{\mathrm{i}\zeta}{\mathrm{i}\zeta-\mathrm{i}\eta} - \frac{\mathrm{i}\zeta}{\mathrm{i}\zeta+\mathrm{i}\eta}\right)H(\eta)\,\mathrm{d}\eta + \theta^+ \\
&\quad + \pi(\mathrm{i}\zeta)H(\zeta) - \pi(\mathrm{i}\zeta)H(-\zeta)\Bigg\} f_2^+(x,\zeta),
\end{aligned} \tag{4.4b}$$

$$\frac{\partial \bar{f}_1^+(x,\zeta)}{\partial t_{2n+1}} = \left\{\sum_{k=0}^{2n} a_k (\mathrm{i}\zeta)^{2n+1-k} + \bar{\theta}^+ - \oint_{-\infty}^{\infty}\left(\frac{\zeta}{\eta-\zeta}+\frac{\zeta}{\eta+\zeta}\right)H(\eta)\,\mathrm{d}\eta \right.$$
$$\left. + \pi(\mathrm{i}\zeta)H(\zeta) - \pi(\mathrm{i}\zeta)H(-\zeta)\right\}\bar{f}_1^+(x,\zeta)$$
$$+ \left\{\sum_{k=0}^{2n} b_k(\mathrm{i}\zeta)^{2n+1-k} + \oint_{-\infty}^{\infty}\frac{\zeta}{\eta-\zeta}H_1(\eta)\,\mathrm{d}\eta + \oint_{-\infty}^{\infty}\frac{\zeta}{\eta+\zeta}H_2(\eta)\,\mathrm{d}\eta \right.$$
$$\left. - \pi(\mathrm{i}\zeta)H_1(\zeta) + \pi(\mathrm{i}\zeta)H_2(-\zeta)\right\}\bar{f}_2^+(x,\zeta), \tag{4.4c}$$

$$\frac{\partial \bar{f}_2^+(x,\zeta)}{\partial t_{2n+1}} = \left\{\sum_{k=0}^{2n} c_k(\mathrm{i}\zeta)^{2n+1-k} - \oint_{-\infty}^{\infty}\frac{\zeta}{\eta-\zeta}H_2(\eta)\,\mathrm{d}\eta - \oint_{-\infty}^{\infty}\frac{\zeta}{\eta+\zeta}H_1(\eta)\,\mathrm{d}\eta \right.$$
$$\left. + \pi(\mathrm{i}\zeta)H_2(\zeta) - \pi(\mathrm{i}\zeta)H_1(-\zeta)\right\}\bar{f}_1^+(x,\zeta)$$
$$+ \left\{\sum_{k=0}^{2n} -a_k(\mathrm{i}\zeta)^{2n+1-k} - \oint_{-\infty}^{\infty}\left(\frac{\mathrm{i}\zeta}{\mathrm{i}\zeta-\mathrm{i}\eta}-\frac{\mathrm{i}\zeta}{\mathrm{i}\zeta+\mathrm{i}\eta}\right)H(\eta)\,\mathrm{d}\eta + \bar{\theta}^+ \right.$$
$$\left. - \pi(\mathrm{i}\zeta)H(\zeta) + \pi(\mathrm{i}\zeta)H(-\zeta)\right\}\bar{f}_2^+(x,\zeta), \tag{4.4d}$$

where the integral $\oint$ is taken as the principal value, and the quantities θ^+, $\bar{\theta}^+$ will be determined in the next section

$$\begin{aligned} &H(\eta) = C(t,\eta)\phi_1(x,t,\eta)\phi_2(x,t,\eta), \\ &H_1(\eta) = C(t,\eta)\phi_1^2(x,t,\eta), \qquad H_2(\eta) = C(t,\eta)\phi_2^2(x,t,\eta). \end{aligned} \tag{4.5}$$

As $x \to -\infty$, we find that the following asymptotics are valid:

$$\begin{aligned} &\oint_{-\infty}^{\infty}\frac{\zeta}{\eta-\zeta}H_1(\eta)\,\mathrm{d}\eta \sim \pi(\mathrm{i}\zeta)C(t,\zeta)a^2(\zeta,t)\mathrm{e}^{-2\mathrm{i}\zeta x}, \\ &\oint_{-\infty}^{\infty}\frac{\zeta}{\eta+\zeta}H_1(\eta)\,\mathrm{d}\eta \sim \pi(\mathrm{i}\zeta)C(t,-\zeta)a^2(-\zeta,t)\mathrm{e}^{2\mathrm{i}\zeta x}, \\ &\oint_{-\infty}^{\infty}\frac{\zeta}{\eta-\zeta}H_2(\eta)\,\mathrm{d}\eta \sim -\pi(\mathrm{i}\zeta)C(t,\zeta)b^2(\zeta,t)\mathrm{e}^{2\mathrm{i}\zeta x}, \\ &\oint_{-\infty}^{\infty}\frac{\zeta}{\eta+\zeta}H_2(\eta)\,\mathrm{d}\eta \sim -\pi(\mathrm{i}\zeta)C(t,-\zeta)b^2(-\zeta,t)\mathrm{e}^{-2\mathrm{i}\zeta x}. \end{aligned} \tag{4.6}$$

Substituting (4.2) into (4.4) and using (4.6), as $x \to -\infty$, we have

$$\frac{\partial S_{22}(\zeta)}{\partial t_{2n+1}} = \left\{-(\mathrm{i}\zeta)^{2n+1} + \theta^+ - \oint_{-\infty}^{\infty}\left(\frac{\zeta}{\eta-\zeta}+\frac{\zeta}{\eta+\zeta}\right)h(\eta)\,\mathrm{d}\eta \right.$$
$$\left. - \pi(\mathrm{i}\zeta)h(\zeta) + \pi(\mathrm{i}\zeta)h(-\zeta)\right\}S_{22}(\zeta)$$
$$- \{2\pi(\mathrm{i}\zeta)h_1(\zeta) - 2\pi(\mathrm{i}\zeta)h_2(-\zeta)\}S_{12}(\zeta),$$

$$\frac{\partial S_{12}(\zeta)}{\partial t_{2n+1}} = \left\{(\mathrm{i}\zeta)^{2n+1} + \theta^+ + \oint_{-\infty}^{\infty}\left(\frac{\zeta}{\eta-\zeta}+\frac{\zeta}{\eta+\zeta}\right)h(\eta)\,\mathrm{d}\eta \right.$$
$$\left. + \pi(\mathrm{i}\zeta)h(\zeta) - \pi(\mathrm{i}\zeta)h(-\zeta)\right\}S_{12}(\zeta),$$

$$
\begin{aligned}
\frac{\partial S_{21}(\zeta)}{\partial t_{2n+1}} &= \bigg\{-(\mathrm{i}\zeta)^{2n+1} + \bar{\theta}^{+} - \oint_{-\infty}^{\infty}\left(\frac{\zeta}{\eta-\zeta} + \frac{\zeta}{\eta+\zeta}\right)h(\eta)\,\mathrm{d}\eta \\
&\quad + \pi(\mathrm{i}\zeta)h(\zeta) - \pi(\mathrm{i}\zeta)h(-\zeta)\bigg\}S_{21}(\zeta), \\
\frac{\partial S_{11}(\zeta)}{\partial t_{2n+1}} &= \bigg\{(\mathrm{i}\zeta)^{2n+1} + \bar{\theta}^{+} + \oint_{-\infty}^{\infty}\left(\frac{\zeta}{\eta-\zeta} + \frac{\zeta}{\eta+\zeta}\right)h(\eta)\,\mathrm{d}\eta \\
&\quad - \pi(\mathrm{i}\zeta)h(\zeta) + \pi(\mathrm{i}\zeta)h(-\zeta)\bigg\}S_{11}(\zeta) \\
&\quad - \{2\pi(\mathrm{i}\zeta)h_2(\zeta) - 2\pi(\mathrm{i}\zeta)h_1(-\zeta)\}S_{21}(\zeta),
\end{aligned}
\tag{4.7}
$$

where

$$
h(\eta) = C(t,\eta)a(\eta,t)b(\eta,t) \qquad h_1(\eta) = C(t,\eta)a^2(\eta,t) \qquad h_2(\eta) = C(t,\eta)b^2(\eta,t).
$$

One can easily see that if $C = 0$ or $a = b = 0$ then the resultant system (4.7) coincides with those equations which appear in the case of the mKdV hierarchy without a source. Using (4.7), we find that the reflection coefficients

$$
R_1(\zeta) = \frac{S_{11}(\zeta)}{S_{21}(\zeta)}, \qquad R_2(\zeta) = \frac{S_{22}(\zeta)}{S_{12}(\zeta)}
\tag{4.8}
$$

satisfy the equation

$$
\begin{aligned}
\frac{\partial R_1(\zeta)}{\partial t_{2n+1}} &= 2\bigg\{(\mathrm{i}\zeta)^{2n+1} + \oint_{-\infty}^{\infty}\left(\frac{\zeta}{\eta-\zeta} + \frac{\zeta}{\eta+\zeta}\right)h(\eta)\,\mathrm{d}\eta - \pi(\mathrm{i}\zeta)h(\zeta) \\
&\quad + \pi(\mathrm{i}\zeta)h(-\zeta)\bigg\}R_1(\zeta) - \{2\pi(\mathrm{i}\zeta)h_2(\zeta) - 2\pi(\mathrm{i}\zeta)h_1(-\zeta)\},
\end{aligned}
\tag{4.9a}
$$

$$
\begin{aligned}
\frac{\partial R_2(\zeta)}{\partial t_{2n+1}} &= 2\bigg\{-(\mathrm{i}\zeta)^{2n+1} - \oint_{-\infty}^{\infty}\left(\frac{\zeta}{\eta-\zeta} + \frac{\zeta}{\eta+\zeta}\right)h(\eta)\,\mathrm{d}\eta - \pi(\mathrm{i}\zeta)h(\zeta) \\
&\quad + \pi(\mathrm{i}\zeta)h(-\zeta)\bigg\}R_2(\zeta) - \{2\pi(\mathrm{i}\zeta)h_1(\zeta) - 2\pi(\mathrm{i}\zeta)h_2(-\zeta)\}.
\end{aligned}
\tag{4.9b}
$$

Then, it follows from (4.9) that the evolution of the reflection coefficients R_1, R_2 is influenced by the integral type of source, which is the integration of the square eigenfunctions belonging to the continuous spectrum of the spectral problem (2.1). For the case $r = q$, there is no discrete eigenvalue for the spectral problem (2.1) if the potential $q = q(x,t)$ tends rather quickly to zero as $|x| \to \infty$. The evolution equations for the reflection coefficients are presented by (4.9), which implies that the mKdVHWS can be solved by the inverse scattering method.

5. Consistency of system (4.7) and equality (4.3)

Let us now verify that system (4.7) is consistent with equality (4.3). First we calculate the parameters θ^{+} and $\bar{\theta}^{+}$. With (4.2) and (4.3) we get

$$
f^{-}(x,\zeta) = -S_{12}(\zeta)\bar{f}^{+}(x,\zeta) + S_{11}(\zeta)f^{+}(x,\zeta)
\tag{5.1a}
$$

$$
\bar{f}^{-}(x,\zeta) = S_{22}(\zeta)\bar{f}^{+}(x,\zeta) - S_{21}(\zeta)f^{+}(x,\zeta).
\tag{5.1b}
$$

According to (2.10) and (4.1a) we can assume

$$
\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = af^{-} - b\bar{f}^{-}.
\tag{5.2}
$$

Using (5.1), (5.2) can be written down as follow:

$$
\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = (aS_{11} + bS_{21})f^{+} - (aS_{12} + bS_{22})\bar{f}^{+}.
\tag{5.3}
$$

According to (5.3) and taking account of (4.5) and (4.1b), as $x \to +\infty$ the following asymptotic is fulfilled:

$$H(\zeta) \sim I(\zeta), \tag{5.4}$$

where

$$\begin{aligned} I(\zeta) = -C(t,\zeta)[S_{11}(\zeta)S_{12}(\zeta)a^2 + (S_{11}(\zeta)S_{22}(\zeta) + S_{12}(\zeta)S_{21}(\zeta))ab + S_{21}(\zeta)S_{22}(\zeta)b^2] \\ = -S_{11}(\zeta)S_{12}(\zeta)h_1(\zeta) - (S_{11}(\zeta)S_{22}(\zeta) \\ + S_{12}(\zeta)S_{21}(\zeta))h(\zeta) - S_{21}(\zeta)S_{22}(\zeta)h_2(\zeta). \end{aligned} \tag{5.5}$$

Using (5.4), (4.4b) and (4.1b) we get

$$\theta^+ = -\mathrm{i}(\mathrm{i}\zeta)^{2n+1} - \oint_{-\infty}^{\infty}\left(\frac{\zeta}{\eta-\zeta} + \frac{\zeta}{\eta+\zeta}\right)I(\eta)\,\mathrm{d}\eta - \pi(\mathrm{i}\zeta)I(\zeta) + \pi(\mathrm{i}\zeta)I(-\zeta). \tag{5.6}$$

Analoguely, using (5.4), (4.4c) and (4.1b) we get

$$\bar{\theta}^+ = \mathrm{i}(\mathrm{i}\zeta)^{2n+1} + \oint_{-\infty}^{\infty}\left(\frac{\zeta}{\eta-\zeta} + \frac{\zeta}{\eta+\zeta}\right)I(\eta)\,\mathrm{d}\eta - \pi(\mathrm{i}\zeta)I(\zeta) + \pi(\mathrm{i}\zeta)I(-\zeta). \tag{5.7}$$

Then, using (5.6) and (5.7), the system (4.7) can be written down as follows:

$$\begin{aligned} \frac{\partial S_{22}(\zeta)}{\partial t} = [-2\mathrm{i}(\mathrm{i}\zeta)^{2n+1} - Q_1(\zeta) - \pi(\mathrm{i}\zeta)M_1(\zeta) - \pi(\mathrm{i}\zeta)M_2(\zeta)]S_{22}(\zeta) \\ - 2\pi(\mathrm{i}\zeta)[h_1(\zeta) - h_2(-\zeta)](\zeta)S_{12}(\zeta), \end{aligned} \tag{5.8a}$$

$$\frac{\partial S_{12}(\zeta)}{\partial t} = [-Q_2(\zeta) - \pi(\mathrm{i}\zeta)M_1(\zeta) + \pi(\mathrm{i}\zeta)M_2(\zeta)]S_{12}(\zeta), \tag{5.8b}$$

$$\frac{\partial S_{21}(\zeta)}{\partial t} = [Q_2(\zeta) - \pi(\mathrm{i}\zeta)M_1(\zeta) + \pi(\mathrm{i}\zeta)M_2(\zeta)]S_{21}(\zeta), \tag{5.8c}$$

$$\begin{aligned} \frac{\partial S_{11}(\zeta)}{\partial t} = -2\pi(\mathrm{i}\zeta)[h_2(\zeta) - h_1(-\zeta)]S_{21}(\zeta) + [2\mathrm{i}(\mathrm{i}\zeta)^{2n} \\ + Q_1(\zeta) - \pi(\mathrm{i}\zeta)M_1(\zeta) - \pi(\mathrm{i}\zeta)M_2(\zeta)]S_{11}(\zeta), \end{aligned} \tag{5.8d}$$

where

$$\begin{aligned} Q_1(\zeta) = \oint_{-\infty}^{\infty}\left(\frac{\zeta}{\eta-\zeta} + \frac{\zeta}{\eta+\zeta}\right)I(\eta)\,\mathrm{d}\eta + \oint_{-\infty}^{\infty}\left(\frac{\zeta}{\eta-\zeta} + \frac{\zeta}{\eta+\zeta}\right)h(\eta)\,\mathrm{d}\eta, \\ Q_2(\zeta) = \oint_{-\infty}^{\infty}\left(\frac{\zeta}{\eta-\zeta} + \frac{\zeta}{\eta+\zeta}\right)I(\eta)\,\mathrm{d}\eta - \oint_{-\infty}^{\infty}\left(\frac{\zeta}{\eta-\zeta} + \frac{\zeta}{\eta+\zeta}\right)h(\eta)\,\mathrm{d}\eta, \\ M_1(\zeta) = I(\zeta) - I(-\zeta), \qquad M_2(\zeta) = h(\zeta) - h(-\zeta). \end{aligned} \tag{5.9}$$

By virtue of (5.8), we have

$$\begin{aligned} \frac{\partial}{\partial t_n}[S_{11}(\zeta)S_{22}(\zeta) - S_{12}(\zeta)S_{21}(\zeta)] = 2\pi(\mathrm{i}\zeta)(M_1(\zeta) - M_2(\zeta))S_{12}(\zeta)S_{21}(\zeta) \\ - 2\pi(\mathrm{i}\zeta)(h_1(\zeta) - h_2(-\zeta))S_{11}(\zeta)S_{12}(\zeta) - 2\pi(\mathrm{i}\zeta)(M_1(\zeta) \\ + M_2(\zeta))S_{11}(\zeta)S_{22}(\zeta) - 2\pi(\mathrm{i}\zeta)(h_2(\zeta) - h_1(-\zeta))S_{21}(\zeta)S_{22}(\zeta). \end{aligned} \tag{5.10}$$

For real function $r = q$, it is known [4] that the eigenfunction $f^-(x,\zeta)$, $\bar{f}^-(x,\zeta)$, $f^+(x,\zeta)$, $\bar{f}^+(x,\zeta)$ defined by (4.1) for (3.4) have symmetry relations

$$\bar{f}^-(x,\zeta) = \begin{pmatrix} -f_2^-(x,t,-\zeta) \\ -f_1^-(x,t,-\zeta) \end{pmatrix}, \qquad \bar{f}^+(x,\zeta) = \begin{pmatrix} f_2^+(x,t,-\zeta) \\ f_1^+(x,t,-\zeta) \end{pmatrix} \tag{5.11a}$$

which imply that that the S_{11}, S_{12}, S_{21}, S_{22} defined by (4.2) satisfy

$$S_{21}(\zeta) = -S_{12}(-\zeta), \qquad S_{22}(\zeta) = -S_{11}(-\zeta). \tag{5.11b}$$

Using (5.11*b*), (5.9) and (5.5), we find that

$$M_1(\zeta) = -S_{11}(\zeta)S_{12}(\zeta)h_1(\zeta) - (S_{11}(\zeta)S_{22}(\zeta) + S_{12}(\zeta)S_{21}(\zeta))h(\zeta) - S_{21}(\zeta)S_{22}(\zeta)h_2(\zeta) + S_{22}(\zeta)S_{21}(\zeta)h_1(-\zeta) + (S_{11}(\zeta)S_{22}(\zeta) + S_{12}(\zeta)S_{21}(\zeta))h(-\zeta) + S_{12}(\zeta)S_{11}(\zeta)h_2(-\zeta). \tag{5.12}$$

In accordance with (5.12), the equality $D(\zeta, t) = S_{11}(\zeta)S_{22}(\zeta) - S_{12}(\zeta)S_{21}(\zeta)$ satisfies the equation

$$\frac{\partial D(\zeta, t)}{\partial t} + 2\pi(\mathrm{i}\zeta)M_1(\zeta)[D(\zeta, t) - 1] = 0.$$

Owing to the condition $D(\zeta, t) = 1$ at $t = 0$ there follows the identity $D(\zeta, t) \equiv 1$ for all $t > 0$. Thus, the consistency of equality (4.3) with system (4.7) is proved.

6. Conclusion

By means of the reduced AKNS eigenvalue problem with $r = q$ which has no discrete eigenvalue, we construct the mKdVHWS. We propose a method to find the evolution equation of the eigenfunction corresponding to the mKdVHWS and further to determine the evolution equation for scattering data, which enables us to solve the mKdVHWS by inverse scattering transformation. Compared with the method for determining the evolution equation for scattering data in [2, 3], our approach is quite natural and simple.

It should be noted that the reduced AKNS spectral problem for $r = -q$ may have a discrete eigenvalue. In this case, the right-hand side of equation (2.9*a*) needs to be supplemented by the sum of square eigenfunctions of (2.9*b*) corresponding to the discrete eigenvalue. We shall show in the forthcoming paper that the mKdV hierarchy with these two kinds of source can also be integrated by inverse scattering transformation.

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